

# VARIATIONAL PRINCIPLE FOR HAMILTONIANS WITH DEGENERATE BOTTOM

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**ABSTRACT.** We consider perturbations of Hamiltonians whose Fourier symbol attains its minimum along a hypersurface. Such operators arise in several domains, like spintronics, theory of superconductivity, or theory of superfluidity. Variational estimates for the number of eigenvalues below the essential spectrum in terms of the perturbation potential are provided. In particular, we provide an elementary proof that negative potentials lead to an infinite discrete spectrum.

We are studying quantum Hamiltonians  $H = H_0 + V$  acting on  $L^2(\mathbb{R}^n)$ ,  $n \geq 2$ , where  $V$  is a potential and  $H_0$  is a self-adjoint (pseudo-differential) operator whose Fourier symbol  $H_0(p)$  attains its minimal value on a certain  $(n-1)$ -dimensional submanifold  $\Gamma$  of  $\mathbb{R}^n$  (surface of extrema). A possible example for  $H_0$  is the Hamiltonian

$$(1) \quad H_0(p) = \Delta + \frac{(|p| - p_0)^2}{2\mu}, \quad \Delta, \mu, p_0 > 0, \quad p \in \mathbb{R}^3,$$

arising in the study of the roton spectrum in liquid helium II [8] and introduced by Landau [9]. Another example can be the three-dimensional Hamiltonian

$$(2) \quad H_0(p) = (p^2 - \mu) \frac{e^{\beta(p^2 - \mu)} + 1}{e^{\beta(p^2 - \mu)} - 1}, \quad \mu, \beta > 0$$

which arised recently in the theory of superconductivity [6, 7]; we refer to the papers cited for the physical meaning of all the constants. Similar operators appear in the study of matrix Hamiltonians related to the spintronics (see below) and in the elasticity theory [5]. A class of operators of the above type were studied in [10] using the Birman-Schwinger approach. In this paper we are going to provide variational estimates for  $H_0 + V$  with localized potentials  $V$  in a possibly simplest form. In particular, we provide an elementary proof for the existence of infinitely many eigenvalues below the essential spectrum for perturbations by negative potentials. Our estimates can be viewed as a generalization

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of the classical result: the existence of a negative eigenvalue for  $-\Delta + V$  in dimensions one and two is guaranteed by the condition  $\int V(x)dx < 0$ .

It seems that the presence of an infinite discrete spectrum in the physics literature in such a setting has been observed first [4] on example of rotationally invariant perturbations of the Rashba Hamiltonian. In the joint papers [1, 2] we gave a rigorous justification for a class of spin-orbit Hamiltonians and rather general potentials, including interactions supported by null sets. The proof was variational and used explicitly the specific properties of two-dimensional systems. Here we develop this idea in a different direction and use the one-dimensional character of the dynamics in the direction transversal to the surface of extrema.

Let us list our assumptions. Below we consider a self-adjoint operator  $H_0 = H_0(-i\nabla)$ , where  $\mathbb{R}^n \ni p \mapsto H_0(p) \in \mathbb{R}$  is a semibounded below continuous function attaining its minimum value  $\min H_0 = m$ . Denote  $\Gamma = \{p \in \mathbb{R}^n : H_0(p) = m\}$ ; we will assume that for some domain  $\Omega \subset \mathbb{R}^n$  the intersection  $S = \Omega \cap \Gamma$  is a smooth  $(n-1)$ -dimensional submanifold of  $\mathbb{R}^n$ ; by  $\omega$  we denote the induced volume form on  $S$ . Without loss of generality we assume that  $\bar{S}$  is compact and orientable (otherwise one can take a smaller  $\Omega$ ). We also suppose that  $H_0$  is at least of  $C^2$  class near  $S$ .

For both the Hamiltonians (1) and (2) one take  $\Omega = \mathbb{R}^3$ . For (1), one has  $m = 0$  and  $S$  is the sphere of radius  $p_0$  centered at the origin. For (2) one has  $m = 2\beta^{-1}$  and  $S$  is the sphere of radius  $\sqrt{\mu}$  centered at the origin.

Consider a real-valued potential  $V \in L^1(\mathbb{R}^n)$ . We will assume that the operator  $H = H_0 + V$  defined as a form sum is self-adjoint with

$$(3) \quad \inf \text{spec}_{\text{ess}}(H_0 + V) = \inf \text{spec}_{\text{ess}} H_0 = m.$$

For both the Hamiltonians (1) and (2) the assumption (3) holds for  $V \in L^{3/2}(\mathbb{R}^3) \cap L^1(\mathbb{R}^3)$ ; indeed, such  $V$  is relatively compact with respect to the Laplacian. As the difference  $(H_0 - \text{Laplacian})$  is infinitely small with respect to the Laplacian,  $V$  is a relatively compact perturbation of  $H_0$  as well.

In what follows we will work in the  $p$ -representation. The operator  $H$  is then associated with the bilinear form

$$\langle f, Hf \rangle = \int_{\mathbb{R}^n} H_0(p) |f(p)|^2 dp + \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \hat{V}(p-p') \overline{f(p)} f(p') dp dp',$$

where  $\hat{V}$  is the Fourier transform of  $V$ ; in our case  $\hat{V}$  is a bounded continuous function due to  $V \in L^1(\mathbb{R}^n)$ . By  $\mathcal{V}$  we denote the operator on  $L^2(S, \omega)$  acting by the rule

$$\mathcal{V}f(s) = \int_S \hat{V}(s-s') f(s') \omega(ds').$$

We note that such an operator already appeared in [10].

**Proposition 1.** *The number of eigenvalues of  $H$  below  $m$  is not less than the number of negative eigenvalues for  $\mathcal{V}$  counting multiplicities.*

*Proof.* Let  $n(s)$  be a unit normal vector to  $S$  at a point  $s \in S$  and depend on  $s$  continuously. For  $r > 0$  consider the map  $L : S \times (-r, r) \rightarrow \mathbb{R}^n$ ,  $(s, t) \mapsto s + tn(s)$ ; we choose  $r$  sufficiently small in order that  $L$  becomes a diffeomorphism between  $S \times (-r, r)$  and  $L(S \times (-r, r))$ . Note that due to the above assumption on  $H_0$  and  $S$  one has  $H(L(s, t)) - m \leq Ct^2$  for  $t \rightarrow 0$  with some  $C > 0$ .

Consider two arbitrary function  $\Psi_1, \Psi_2 \in L^2(S, \omega)$ . Take an arbitrary function  $\varphi \in C_0^\infty(\mathbb{R})$  with  $\int \varphi = 1$  and  $\varepsilon > 0$ . Consider functions  $f_j^\varepsilon \in L^2(\mathbb{R}^n)$  given by

$$(4) \quad f_j^\varepsilon(p) = \begin{cases} \frac{1}{\varepsilon} \varphi\left(\frac{t}{\varepsilon}\right) \Psi_j(s), & p = L(s, t), \quad (s, t) \in S \times (-r, r), \\ 0 & \text{otherwise.} \end{cases}$$

Clearly,

$$\begin{aligned} \langle f_1^\varepsilon, (H - m)f_2^\varepsilon \rangle &= \int_{\mathbb{R}^n} \overline{f_1^\varepsilon(p)} (H_0(p) - m) f_2^\varepsilon(p) dp \\ &\quad + \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \hat{V}(p - p') \overline{f_1^\varepsilon(p)} f_2^\varepsilon(p') dp dp'. \end{aligned}$$

One has  $dp = \rho(s, t)\omega(ds)dt$  with  $\rho(s, t) = 1 + O(t)$  for  $t \rightarrow 0$  uniformly in  $s \in S$ , hence

$$\begin{aligned} &\left| \int_{\mathbb{R}^n} (H_0(p) - m) \overline{f_1^\varepsilon(p)} f_1^\varepsilon(p) dp \right| \\ &= \left| \frac{1}{\varepsilon^2} \int_{-r}^r \int_S (H(L(s, t)) - m) \left| \varphi\left(\frac{t}{\varepsilon}\right) \right|^2 \overline{\Psi_1(s)} \Psi_2(s) \rho(s, t) \omega(ds) dt \right| \\ &\leq C \left| \frac{1}{\varepsilon^2} \int_{-r}^r \int_S t^2 \left| \varphi\left(\frac{t}{\varepsilon}\right) \right|^2 \overline{\Psi_1(s)} \Psi_2(s) \rho(s, t) \omega(ds) dt \right| \\ &\leq C\varepsilon \left| \int_{-r/\varepsilon}^{r/\varepsilon} \int_S t^2 \left| \varphi(t) \right|^2 \overline{\Psi_1(s)} \Psi_2(s) \rho(s, \varepsilon t) \omega(ds) dt \right| = O(\varepsilon). \end{aligned}$$

On the other hand, for any bounded continuous function  $v : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}$  one has

$$\begin{aligned}
 (5) \quad & \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} v(p, p') \overline{f_1^\varepsilon(p)} f_2^\varepsilon(p') dp dp' \\
 &= \frac{1}{\varepsilon^2} \int_{-r}^r \int_{-r}^r \int_S \int_S v(L(s, t), L(s', t')) \overline{\varphi\left(\frac{t}{\varepsilon}\right)} \varphi\left(\frac{t'}{\varepsilon}\right) \\
 &\quad \times \overline{\Psi_1(s)} \Psi_2(s') \rho(s, t) \rho(s', t') \omega(ds) \omega(ds') dt dt' \\
 &\quad \int_{-r/\varepsilon}^{r/\varepsilon} \int_{-r/\varepsilon}^{r/\varepsilon} \int_S \int_S v(L(s, \varepsilon t), L(s', \varepsilon t')) \overline{\varphi(t)} \varphi(t') \\
 &\quad \times \overline{\Psi_1(s)} \Psi_2(s') \rho(s, \varepsilon t) \rho(s', \varepsilon t') \omega(ds) \omega(ds') dt dt' =: I(\varepsilon).
 \end{aligned}$$

Due to the obvious estimate

$$\begin{aligned}
 & \left| \int_S \int_S v(L(s, \varepsilon t), L(s', \varepsilon t')) \right. \\
 & \quad \times \overline{\Psi_1(s)} \Psi_2(s') \rho(s, \varepsilon t) \rho(s', \varepsilon t') \omega(ds) \omega(ds') \Big| \\
 & \leq \tilde{C} \int_S |\Psi_1(s)| \omega(ds) \int_S |\Psi_2(s)| \omega(ds)
 \end{aligned}$$

with  $\tilde{C} = \sup_{p, p' \in \mathbb{R}^n} |v(p, p')| \sup_{(s, t) \in S \times (-r, r)} |\rho(s, t)|$ , one has, by the Lebesgue dominated convergence,

$$\begin{aligned}
 (6) \quad \lim_{\varepsilon \rightarrow 0} I(\varepsilon) &= \int_S \int_S v(L(s, 0), L(s', 0)) \overline{\Psi_1(s)} \Psi_2(s') \omega(ds) \omega(ds') \\
 &= \int_S \int_S v(s, s') \overline{\Psi_1(s)} \Psi_2(s') \omega(ds) \omega(ds').
 \end{aligned}$$

Taking  $v(p, p') = \hat{V}(p - p')$ , we have shown that for any  $\Psi_1, \Psi_2 \in L^2(S, \omega)$  and the functions  $f_1^\varepsilon, f_2^\varepsilon$  given by (4) one has

$$\lim_{\varepsilon \rightarrow 0} \langle f_1^\varepsilon, (H - m) f_2^\varepsilon \rangle = \langle \Psi_1, \mathcal{V} \Psi_2 \rangle.$$

Assume now that  $\mathcal{V}$  has  $N$  negative eigenvalues  $E_1, \dots, E_N$  and let  $\Psi_1, \dots, \Psi_N$  be the corresponding normalized eigenfunctions orthogonal to each other. Consider the functions  $f_j^\varepsilon, j = 1, \dots, N$ , given by (4). Then, by the above arguments, the matrix  $h(\varepsilon) = (\langle f_j^\varepsilon, (H - m) f_k^\varepsilon \rangle)$  converges to  $\text{diag}(E_1, \dots, E_N)$ . In particular,  $h(\varepsilon)$  is negative definite for sufficiently small  $\varepsilon$ , which means, by the variational principle, that  $H$  has at least  $N$  eigenvalues below  $m$ .  $\square$

Due to the obvious estimate

$$\int_S \int_S |\hat{V}(s - s')|^2 \omega(ds) \omega(ds') < \infty$$

$\mathcal{V}$  is a Hilbert-Schmidt operator and hence compact, which implies  $\text{spec}_{\text{ess}} \mathcal{V} = \{0\}$ .

**Proposition 2.** *If  $V \leq 0$  and  $V \not\equiv 0$ , then the discrete spectrum of  $\mathcal{V}$  consists of an infinite sequence of negative eigenvalues converging to 0, and 0 is not an eigenvalue.*

*Proof.* Let  $f \in L^2(S, \omega)$ . One has

$$\begin{aligned} \langle f, \mathcal{V}f \rangle &= \int_S \int_S \hat{V}(s-s') \overline{f(s)} f(s') \omega(ds) \omega(ds') \\ &= \frac{1}{(2\pi)^{n/2}} \int_S \int_S \int_{\mathbb{R}} V(x) e^{i\langle s'-s, x \rangle} \overline{f(s)} f(s') dx \omega(ds) \omega(ds') \\ &= \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}} V(x) |g(x)|^2 dx \leq 0 \end{aligned}$$

with

$$(7) \quad g(x) := \int_S f(s) e^{-i\langle s, x \rangle} \omega(ds).$$

Therefore,  $\text{spec } \mathcal{V} \subset (-\infty, 0]$ .

Assume that  $\langle f, \mathcal{V}f \rangle = 0$  for some  $f$ . The function  $g$  in (7) is analytic as the Fourier transform of the compactly supported distribution  $(2\pi)^{n/2} f(s) \delta_S(s)$ , where  $\delta_S$  is the delta measure concentrated on  $S$ . To have  $\langle f, \mathcal{V}f \rangle = 0$  the function  $g$  must vanish on a set of non-zero Lebesgue measure (the support of  $V$ ) and hence vanish everywhere. As the Fourier transform is a bijection on the set of tempered distributions, this means  $f = 0$ . Therefore, 0 cannot be an eigenvalue of  $\mathcal{V}$ , and it remains to recall that  $\mathcal{V}$  is compact.  $\square$

Combining propositions 1 and 2 one arrives at

**Corollary 3.** *If  $V \leq 0$  and  $V \not\equiv 0$ , then  $H$  has infinitely many eigenvalues below the essential spectrum.*

If the condition  $V \leq 0$  does not hold, one still can try to estimate the number of negative eigenvalues for  $\mathcal{V}$  using the values of the Fourier transform at some points. Due to  $\text{spec}_{\text{ess}} \mathcal{V} = \{0\}$  the number of negative eigenvalues for  $\mathcal{V}$  can be estimated using the variational principle as well.

**Proposition 4.** *Let  $N \in \mathbb{N}$ . Assume that there exist points  $s_j \in S$ ,  $j = 1, \dots, N$ , such that the matrix  $(\hat{V}(s_j - s_k))$  is negative definite, then  $V$  has at least  $N$  negative eigenvalues and hence  $H$  has at least  $N$  eigenvalues below  $m$ .*

*Proof.* Fix some neighborhoods  $S_j \subset S$  of  $s_j$  such that there exists diffeomorphisms  $J_j : B \rightarrow S_j$ , where  $B$  is the unit ball centered at the origin in  $\mathbb{R}^{n-1}$ . Without loss of generality we assume  $J_j(0) = s_j$ . Let us take a function  $\varphi \in C_0^\infty(\mathbb{R}^{n-1})$  with  $D_j(0) \int \varphi = 1$ , where  $D_j$  is the Jacobian for  $J_j$ . Denote  $\Psi_j^\varepsilon(s) = \varepsilon^{1-n} \varphi(\varepsilon^{-1} J_j^{-1}(s)) \chi_B(J_j^{-1}(s))$ ; clearly,

$\Psi_j^\varepsilon \in L^2(S, \omega)$ . One has

$$\begin{aligned} \langle \Psi_j^\varepsilon, \mathcal{V} \Psi_k^\varepsilon \rangle &= \int_S \int_S \overline{\Psi_j^\varepsilon(s)} \hat{V}(s-s') \Psi_k^\varepsilon(s') \omega(ds) \omega(ds') \\ &= \varepsilon^{2-2n} \int_B \int_B \overline{\varphi(u/\varepsilon)} \hat{V}(J_j(u) - J_k(u')) \varphi(u'/\varepsilon) D_j(u) D_k(u') du du' \\ &\quad \int_{B/\varepsilon} \int_{B/\varepsilon} \overline{\varphi(u)} \hat{V}(J_j(\varepsilon u) - J_k(\varepsilon u')) \varphi(u') D_j(\varepsilon u) D_k(\varepsilon u') du du' \\ &\xrightarrow{\varepsilon \rightarrow 0} \hat{V}(s_j - s_k). \end{aligned}$$

Therefore, the matrix  $(\langle \Psi_j^\varepsilon, \mathcal{V} \Psi_k^\varepsilon \rangle)$  is negative definite for small  $\varepsilon$ . The rest follows from the variational principle and proposition 1.  $\square$

Taking  $N = 1$  in proposition 4 we obtain a simple condition resembling that for perturbations of the Laplacian in one and two dimensions.

**Corollary 5.** *If  $\int V(x)dx < 0$ , then  $H$  has at least one eigenvalue below  $m$ .*

We note that corollary 3 can be also obtained from proposition 4 because for  $V \leq 0$  and  $V \not\equiv 0$  the matrix  $(\hat{V}(s_j - s_k))$  is negative definite for any choice and any number of mutually distinct points  $s_j \in \mathbb{R}^n$  by the Bochner theorem.

The above constructions can be also applied to a class of matrix Hamiltonians. Namely, consider an operator  $H_0$  acting in  $L^2(\mathbb{R}^n) \otimes \mathbb{C}^d$  whose Fourier symbol in the multiplication by a  $d \times d$  Hermitian matrix  $H_0(p)$ . Then there exist unitary matrices  $U(p)$ ,  $p \in \mathbb{R}^n$ , and real-valued continuous functions  $p \mapsto \lambda_1(p)$ ,  $\dots$ ,  $p \mapsto \lambda_d(p)$  with  $\lambda_1(p) \leq \dots \leq \lambda_d(p)$  such that

$$H_0(p) = U(p) \operatorname{diag}(\lambda_1(p), \dots, \lambda_d(p)) U^*(p).$$

We assume that  $\lambda_1(p)$  satisfies the same conditions as the symbol  $H_0(p)$  in the scalar case. We will use the same notation; in particular,  $\min \lambda_1(p) = \inf \operatorname{spec} H_0 = m$ .

A class of such matrix operators is delivered by spin-orbit Hamiltonians [12] acting in  $L^2(\mathbb{R}^2) \otimes \mathbb{C}^2$  and given by the matrices

$$(8) \quad H_0(p) = \begin{pmatrix} p^2 & a(p) \\ a(p) & p^2 \end{pmatrix}$$

with some linear functions  $a$ . In particular, the case  $a(p) = \alpha(p_2 + ip_1)$  corresponds to the Rashba Hamiltonian [3], and  $a(p) = -\alpha(p_1 + ip_2)$  gives the Dresselhaus Hamiltonian [11]; in both cases  $\alpha$  is a non-zero constant. Here one has  $\lambda_1(p) = p^2 - |a(p)|$ , and the minimum  $-\alpha^2/4$  is attained at the circle  $|p| = |\alpha|/2$ .

Again consider a *scalar* real-valued potential  $V \in L^1(\mathbb{R}^n)$ . Assume that the operator  $H_0 + V$  defined through the form sum is self-adjoint and that  $\inf \operatorname{spec}_{\operatorname{ess}}(H_0 + V) = \inf \operatorname{spec}_{\operatorname{ess}} H_0 = m$ . The preservation of

the essential spectrum for the above Rashba and Dresselhaus Hamiltonians is guaranteed, e.g. for  $V \in L^1(\mathbb{R}^2) \cap L^2(\mathbb{R}^2)$ , which is achieved by comparison with the two-dimensional Laplacian.

Also in this case we can prove an analogue of corollary 3.

**Proposition 6.** *Let  $V \leq 0$  and  $V \not\equiv 0$ , then the matrix Hamiltonian  $H$  has infinitely many eigenvalues below  $m$ .*

*Proof.* The proof follows the construction in the proof of proposition 1. Consider the vector  $h = (1, 0, \dots, 0)^T \in \mathbb{C}^d$ , and for  $\varepsilon > 0$  denote  $F_j^\varepsilon(p) = U(p)f_j^\varepsilon(p)h$  with the functions  $f_j^\varepsilon$  from (4). Then one has

$$\begin{aligned} \langle F_j^\varepsilon, (H - m)F_k^\varepsilon \rangle &= \int_{\mathbb{R}^n} \lambda_1(p) \overline{f_j^\varepsilon(p)} f_k^\varepsilon(p) dp \\ &\quad + \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \hat{V}(p - p') \langle U(p)h, U(p')h \rangle \overline{f_j^\varepsilon(p)} f_k^\varepsilon(p') dp dp'. \end{aligned}$$

By (5) and (6), there holds

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} \langle F_j^\varepsilon, (H - m)F_k^\varepsilon \rangle \\ = \int_S \int_S \hat{V}(s - s') \langle U(s)h, U(s')h \rangle \overline{\Psi_1(s)} \Psi_2(s') \omega(ds) \omega(ds'). \end{aligned}$$

By the arguments of proposition 1, the number of eigenvalues of  $H$  below  $m$  is not less than the number of negative eigenvalues of the operator  $\mathcal{U}$  acting on  $L^2(S, \omega)$  and given by

$$\mathcal{U}f(s) = \int_S \hat{V}(s - s') \langle U(s)h, U(s')h \rangle f(s') ds'.$$

Again, by

$$\begin{aligned} \int_S \int_S \left| \hat{V}(s - s') \langle U(s)h, U(s')h \rangle \right|^2 \omega(ds) \omega(ds') \\ \leq \int_S \int_S |\hat{V}(s - s')|^2 \omega(ds) \omega(ds') < \infty, \end{aligned}$$

$\mathcal{U}$  is a compact operator. Let us show that all eigenvalues of  $\mathcal{U}$  are negative (this, like in proposition 2, will mean that  $\mathcal{U}$  has an infinite number of negative eigenvalues). For  $f \in L^2(S, \omega)$  one has

$$\begin{aligned} (9) \quad \langle f, \mathcal{U}f \rangle &= \int_S \int_S \hat{V}(s - s') \langle U(s)h, U(s')h \rangle \overline{f(s)} f(s') \omega(ds) \omega(ds') \\ &= \int_S \int_S \int_{\mathbb{R}} \frac{V(x) e^{i\langle s' - s, x \rangle}}{(2\pi)^{n/2}} \langle U(s)h, U(s')h \rangle \overline{f(s)} f(s') dx \omega(ds) \omega(ds') \\ &= \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}} V(x) |g(x)|^2 dx \leq 0 \end{aligned}$$

with

$$g(x) := \int_S U(s) f(s) h e^{-i\langle s, x \rangle} \omega(ds).$$

It remains to show that 0 is not an eigenvalue of  $\mathcal{U}$ . Assuming  $\mathcal{U}f = 0$  we obtain from (9) that  $g$  vanishes on the support of  $V$  having non-zero Lebesgue measure. As  $g$  is again the Fourier transform on a compactly supported distribution and hence analytic, it must vanish everywhere, which means that the vector function  $s \mapsto U(s)f(s)h$  is zero a.e. As the matrix  $U(s)$  is unitary for any  $s$ , this means  $f = 0$ .  $\square$

We note that proposition 4 and corollary 3 for the Rashba and Dresselhaus Hamiltonians were shown in [1] using test functions of a different type. The above analysis can be extended to the case when the perturbation  $V$  is not a potential, but a measure with some regularity conditions. For the Hamiltonians (8) one can still prove the infiniteness of the discrete spectrum for perturbations by negative measures supported by curves [2].

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